Distributed and Parallel Algorithms for Set Cover Problems with Small Neighborhood Covers

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Set Cover

- Input: Set system $\langle E, \mathcal{S} \rangle$
- $E : e_1, e_2, e_3, \ldots, e_m$
- $\mathcal{S} : \underset{c(S_1)}{\{\ldots\}}, \underset{c(S_2)}{\{\ldots\}}, \underset{c(S_3)}{\{\ldots\}}, \ldots, \underset{c(S_n)}{\{\ldots\}}$
- Output: $\mathcal{R} \subseteq \mathcal{S}$ having minimum cost such that all the $e \in E$ are covered

Prior Work

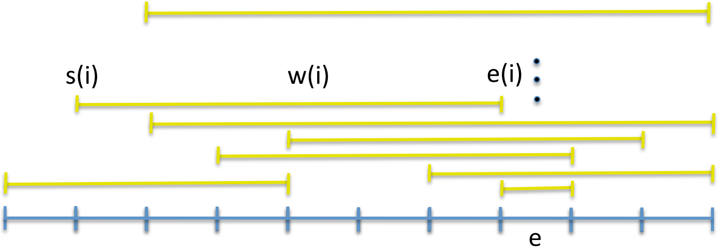
- Sequential setting
  - $O(\log \Delta)$ approximation ratio, Δ is the maximum cardinality of the sets in $\mathcal{S}$
  - f approximation ratio, where f is the *frequency parameter* which is the maximum number of sets of $\mathcal{S}$ that any element belongs to
  - $\Omega(\log m)$-inapproximable

- Parallel setting
  - [RV '98]$O(\log m)$ approximation ratio
  - [KVY '94]$O(f + \epsilon)$ approximation ratio

- Distributed setting
  - [Kuhn et al. '06]$O(\log \Delta)$ approximation ratio
  - [KY '11]f approximation ratio
  - both the above algorithms run in $O(\log m)$ communication rounds

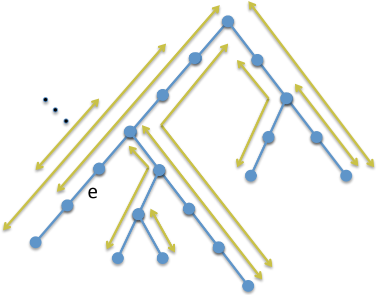
Interval Cover

- Non constant f but
  - constant factor by Primal Dual
  - optimal by DP



Tree Cover

- Non constant f but
  - constant factor by Primal Dual
  - optimal by DP



Other Problems with non constant f

- Priority Interval Cover
  - optimal by Primal Dual
- Bag Interval Cover (NP Hard)
  - constant factor by Primal Dual

Our contributions

- Introduction of SNC property for a class of set cover problems
  - A wide range of problems fall under this framework

- For set cover problems with non-constant f but satisfying the SNC property, we give constant factor approximation algorithms in
  - sequential setting,
  - parallel setting, and
  - distributed setting

SNC property

- E : $e_1, e_2, e_3, \ldots, e_m$
- $\mathcal{S}$: $\{\ldots\}, \{\ldots\}, \{\ldots\}, \ldots, \{\ldots\}$

- The *neighborhood* of an element $e \in E$ is the set of all the elements with which it shares some set in $\mathcal{S}$
  - $\mathcal{S}$: $\{e_1, e_2\}, \{e_1, e_2, e_3\}, \{e_1, e_4, e_6\}, \{e_2, e_5\}$
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  - neighborhood(e_1): $\{e_1, e_2, e_3, e_4, e_6\}$
  - $\mathcal{S}$: $\{e_1, e_2\}, \{e_1, e_2, e_3\}, \{e_1, e_4, e_6\}, \{e_2, e_5\}$
  - neighborhood(e_2): $\{e_1, e_2, e_3, e_5\}$

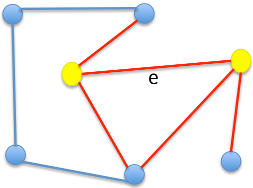
- An element is a τ-SNC element if its neighborhood can be covered by atmost τ sets
  - $e_1 \rightarrow$ 3-SNC
  - $e_2 \rightarrow$ 2-SNC

SNC property

- For any subset X, the set system restricted to X is defined as $\langle X, \mathcal{S}' \rangle$ where $\mathcal{S}' = \{S \cap X \ : \ S \in \mathcal{S}\}$
- For a τ, if any restriction of the set system contains atleast one τ-SNC element, we call the set system as τ-SNC set system

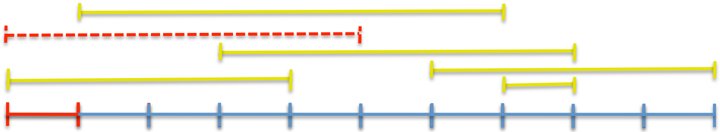
Examples: Vertex Cover : 2-SNC

- Each edge (element) is incident upon 2 vertices (sets)



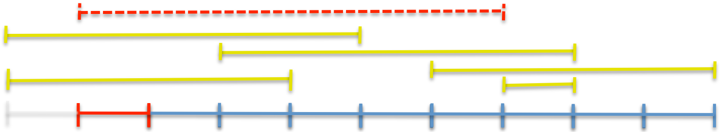
Examples: Interval Cover : 1-SNC

- The leftmost element is a 1-SNC element – Its neighborhood can be covered by the interval spanning it and extending most towards the right



Examples: Interval Cover : 1-SNC

- After removing this 1-SNC element, we can find another 1-SNC element



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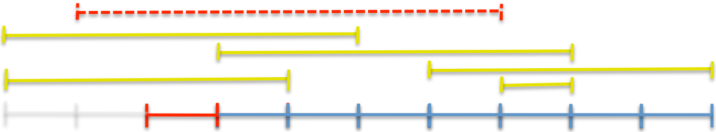
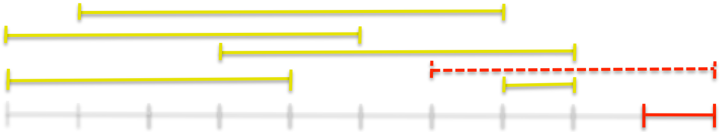


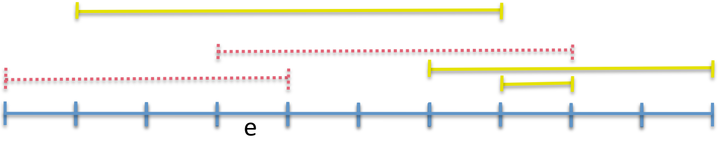
Examples: Interval Cover : 1-SNC

- Only after we have removed the leftmost 1-SNC element, we find a new 1-SNC element



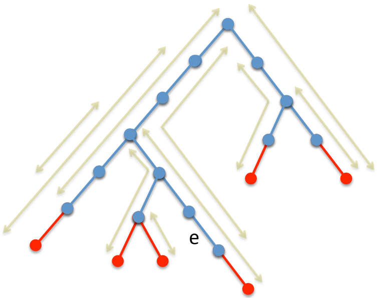
Examples: Interval Cover : 2-SNC

- Can be viewed as a 2-SNC set system as well
- The neighborhood of any timeslot (element) can be covered by 2 intervals (sets) - the one extending most towards the left and the one extending most towards the right



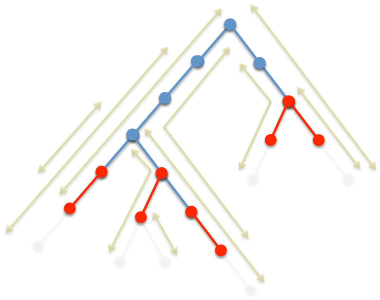
Examples: Tree Cover : 1-SNC

- All the leaves are 1-SNC elements – Their neighborhood can be covered by the interval extending most towards the root



Examples: Tree Cover : 1-SNC

- Any restriction of a tree is a tree in itself. After removing all the 1-SNC element, we can find other 1-SNC elements



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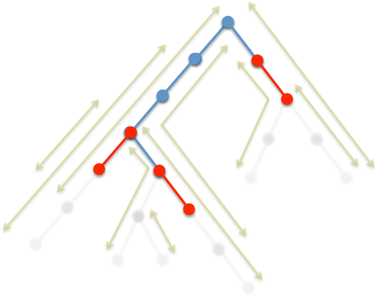

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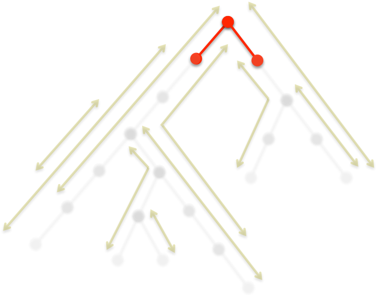
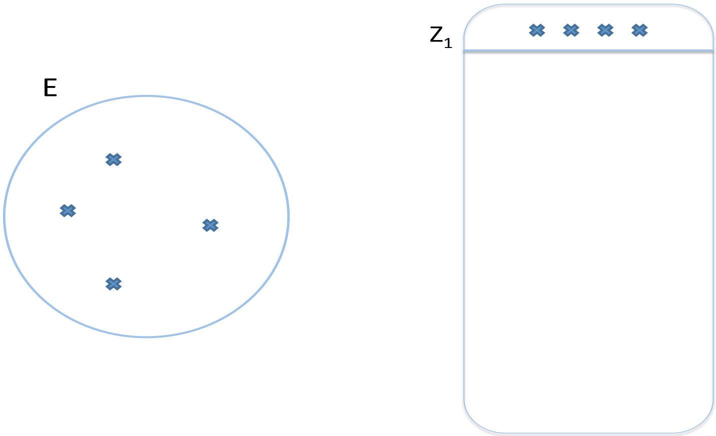


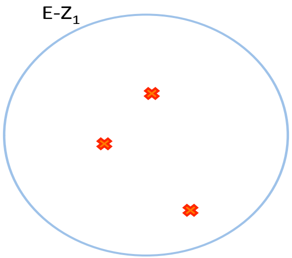
Layer Decomposition

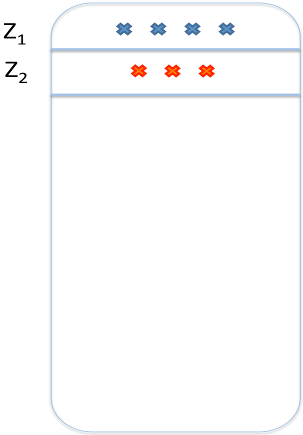
- In the given set system, find all τ-SNC elements. Put them in layer Z_1



Layer Decomposition

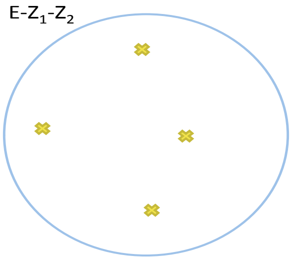
- Restrict the set system to $E - Z_1$. Find all τ-SNC elements in the restriction

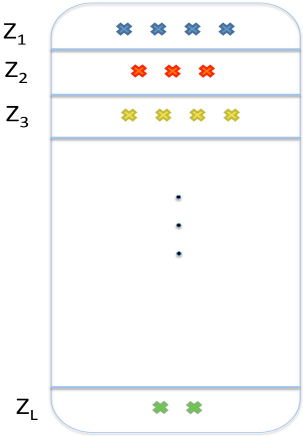




Layer Decomposition

- Continue till all the elements belong to some layer Z_k





Layer Decomposition: Length

- Interval Cover
  - 1-SNC $\rightarrow L = \Omega(m)$
  - 2-SNC $\rightarrow L = 1$

Layer Decomposition: Length

- Interval Cover
  - 1-SNC $\rightarrow L = \Omega(m)$
  - 2-SNC $\rightarrow L = 1$

- Tree Cover
  - 1-SNC $\rightarrow L = \Omega(m)$
  - *Theorem: For a 2-SNC tree cover system, the number of layers* $L = O(\log m)$

Layer Decomposition: Length

There exists a procedure for computing the layer decomposition of a given τ-SNC set system.

- In the sequential setting, it can be implemented in polynomial time.
- In the distributed setting, it can be implemented in $O(L)$ communication rounds.
- In the parallel setting, the algorithm takes L iterations each of which can be implemented in NC.

- For parallel and distributed setting the decomposition length (L) should be $O(\log m)$

Our Results

- τ sequential approximation algorithm for any τ-SNC set system
- $8\tau^2$ parallel approximation algorithm for a τ-SNC set system of logarithmic length
- τ distributed approximation algorithm for a τ-SNC set system of logarithmic length

Primal Dual Framework

Primal

$$\min \quad \sum_{S \in \mathcal{S}} x(S) \cdot w(S)$$

$$\begin{aligned} \sum_{S \in \mathcal{S} : e \in S} x(S) &\geq 1 \quad (\forall e \in E) \\ x(S) &\geq 0 \quad (\forall S \in \mathcal{S}) \end{aligned}$$

Dual

$$\max \quad \sum_{e \in E} \alpha(e)$$

$$\begin{aligned} \sum_{e \in S} \alpha(e) &\leq w(S) \quad (\forall S \in \mathcal{S}) \\ \alpha(e) &\geq 0 \quad (\forall e \in E) \end{aligned}$$

Primal Dual Framework

- A primal-dual algorithm would generally have
  - Forward Phase
  - Reverse Delete Phase

- *Forward Phase*
  - Increase the dual variables
  - If some constraint becomes tight, pick the variable corresponding to that constraint in the primal solution
  - Output $\langle A, \alpha \rangle$, where A and α are the primal and dual solutions respectively

- *Reverse Delete Phase*
  - Take the primal solution A produced in the forward phase and *drop* redundant items from it
  - Output $\langle B, \alpha \rangle$, where $B \subseteq A$

Primal Dual Framework: Approx. Complementary Slackness Conditions

- We say that the pair $\langle A, \alpha \rangle$ is λ-*maximal*, if for any $S \in \mathcal{A}$, the corresponding dual constraint is approximately tight:

$$\sum_{e \in S} \alpha(e) \geq \lambda \cdot w(S)$$

- We say that the pair $\langle B, \alpha \rangle$ satisfies the primal slackness conditions approximately, if for any $e \in E$, if $\alpha(e) > 0$ then

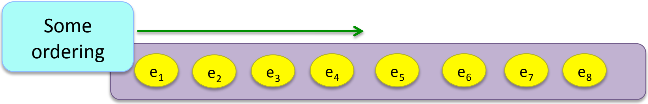
$$|\{S \in B \ : \ S \text{ covers } e\}| \leq \mu.$$

- The overall factor achieved is $\frac{1}{\lambda} \cdot \mu$

Primal Dual Framework: Forward Phase

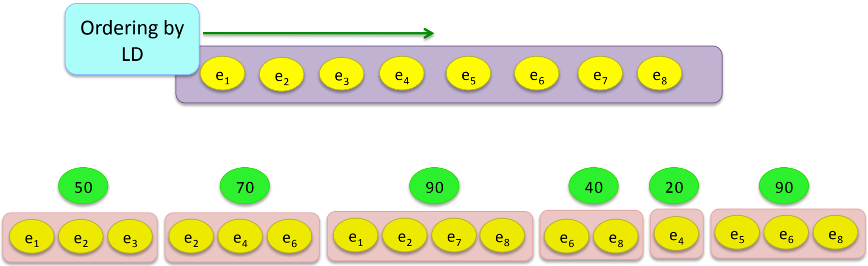
Dual

$$\begin{aligned} \max \quad & \sum_{e \in E} \alpha(e) & & \\ & \sum_{e \in S} \alpha(e) \leq w(S) & & (\forall S \in \mathcal{S}) \\ & \alpha(e) \geq 0 & & (\forall e \in E) \end{aligned}$$



$O(\tau)$ sequential algorithm: Forward Phase

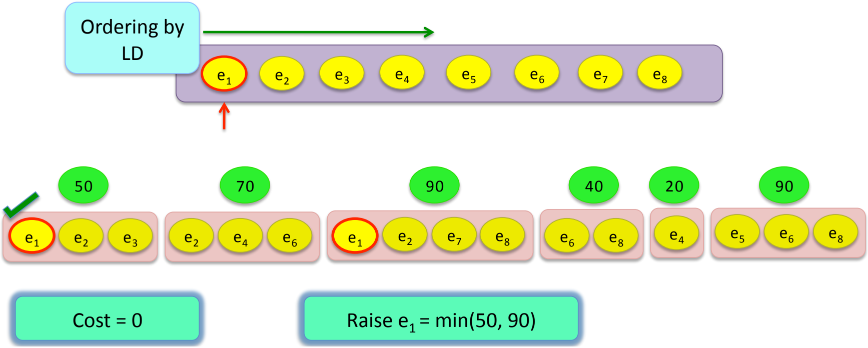
- Arrange the elements according to layer decomposition order
- Scan them from left to right



$\sum_{e \in S} \alpha(e) \leq w(S) \quad (\forall S \in \mathcal{S})$

$O(\tau)$ sequential algorithm: Forward Phase

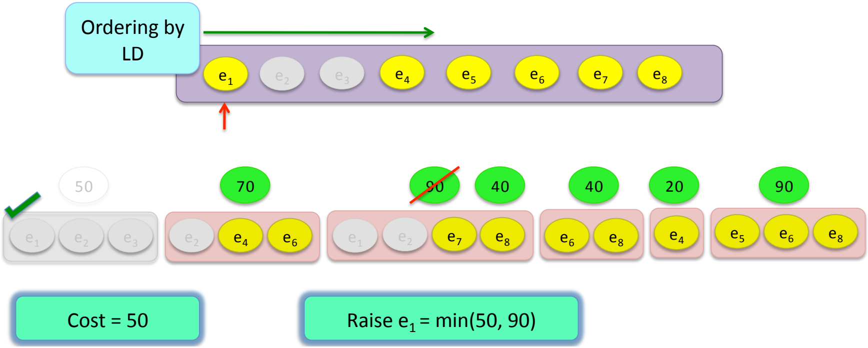
- Raise the first uncovered element to the maximum possible cost



$\sum_{e \in S} \alpha(e) \leq w(S) \quad (\forall S \in \mathcal{S})$

$O(\tau)$ sequential algorithm: Forward Phase

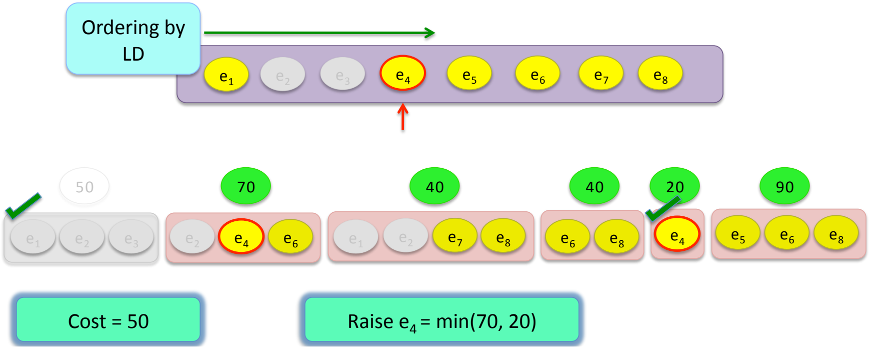
- Include the *tight* set in the solution
- Remove the elements that got *freely* covered by this picked up set



$\sum_{e \in S} \alpha(e) \leq w(S) \quad (\forall S \in \mathcal{S})$

$O(\tau)$ sequential algorithm: Forward Phase

- Look for the next uncovered element in the ordering and repeat the process



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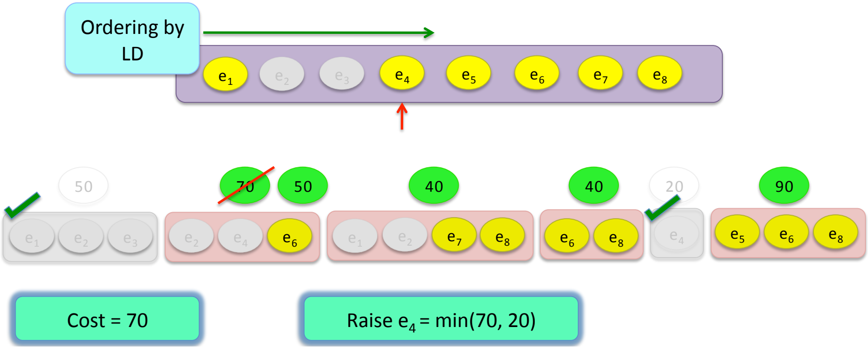

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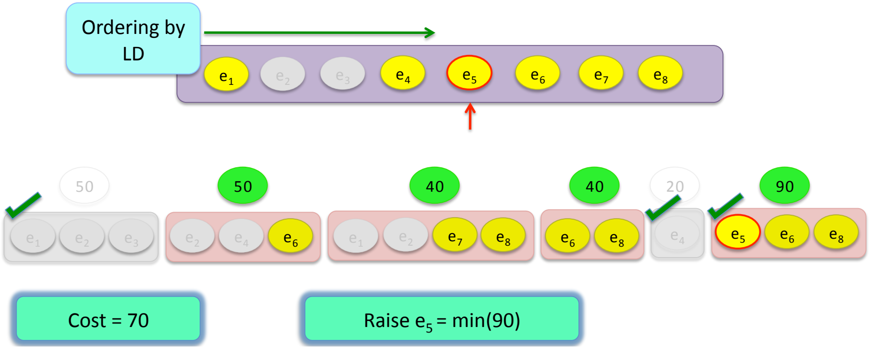

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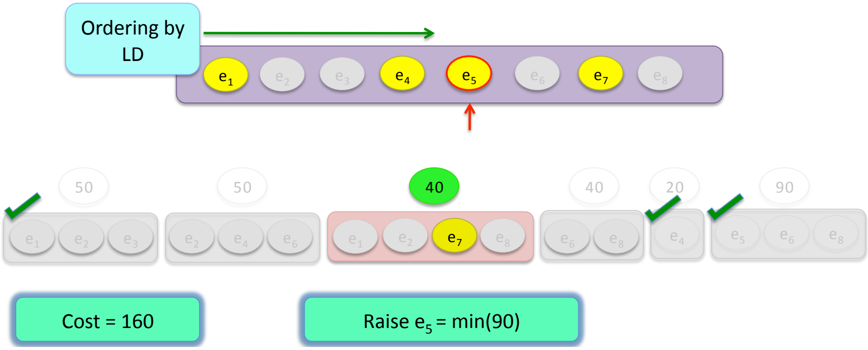

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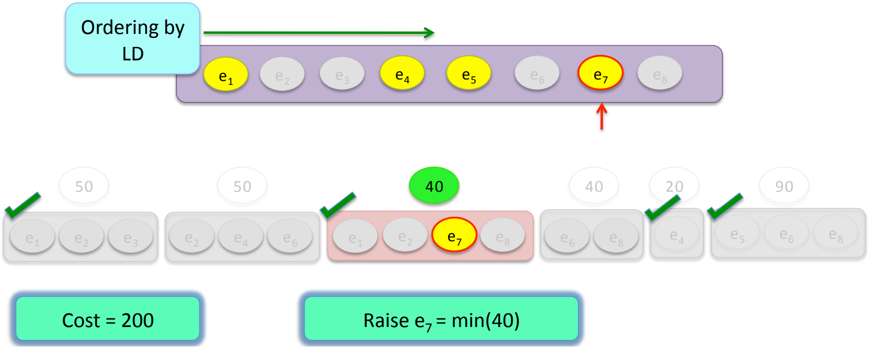
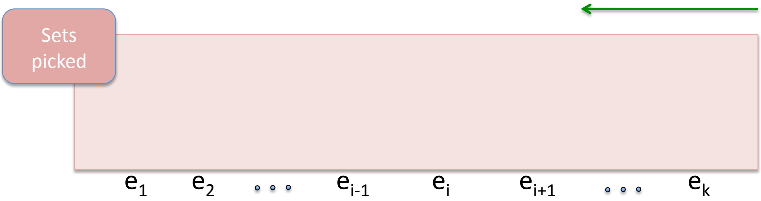


$\sum_{e \in S} \alpha(e) \leq w(S) \quad (\forall S \in \mathcal{S})$

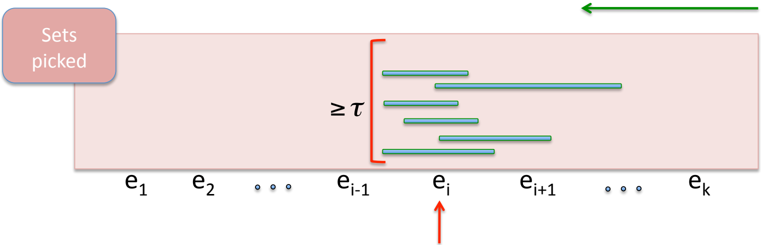
$O(\tau)$ sequential algorithm: Reverse Delete Phase

- Only look at the elements *raised* in the forward phase
- Arrange them in the order they were raised and scan them in the reverse direction



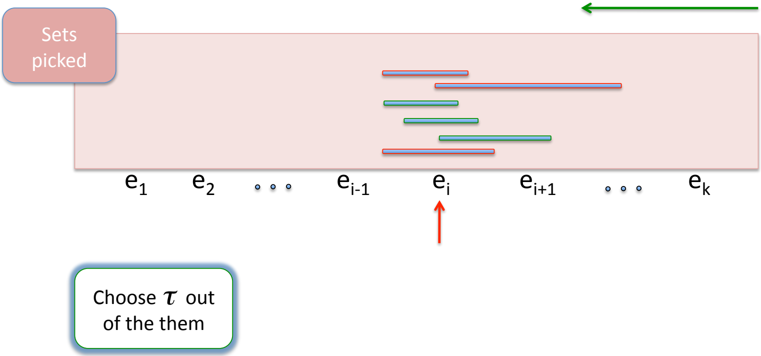
$O(\tau)$ sequential algorithm: Reverse Delete Phase

- It's possible that more than τ sets are spanning an element raised in the forward phase



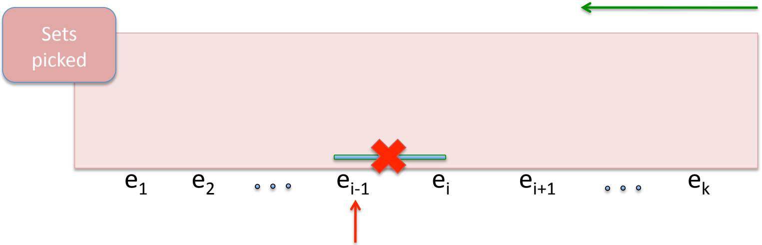
$O(\tau)$ sequential algorithm: Reverse Delete Phase

- Bad!! *Compress them!!*
- Choose at most τ out of them



$O(\tau)$ sequential algorithm: Reverse Delete Phase

- After the compression, can we still guarantee coverage?
- Elements to its left are not dependent on it for their coverage.
- No set picked up in $i-1$ iterations can span e_i else e_i wouldn't have been raised in the forward phase



$O(\tau)$ sequential algorithm: Reverse Delete Phase

- What about the elements to its right?
- e_i is τ SNC in the restriction to its right (Layer decomposition)

- $\lambda = 1$, and
- $\mu = \tau$. Hence, we have $O(\tau)$ approximation algorithm

Parallel Algorithm: Comparison with [KVY '94]

Constant f sequential setting

- Raise one variable at a time
- Produce maximal solutions ($\lambda = 1$)
- # of iterations $= \Omega(m)$

$[KVY'94]$ parallel setting

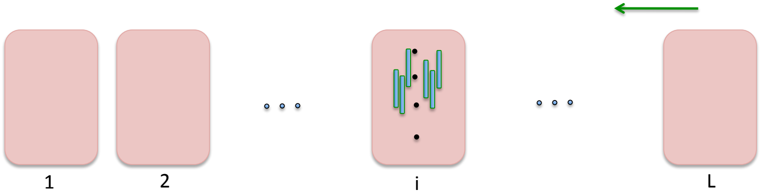
- Raise multiple variables simultaneously
- Produce near maximal solutions ($\lambda = 1 - \epsilon$)
- # of iterations $= O(\frac{1}{\epsilon} f \log m)$

Our Results in Parallel setting

- Go according to layer decomposition. Inside each layer,
  - Raise multiple variables simultaneously
  - Produce (1/8) maximal solutions (worse than [KVY])
  - # of iterations $O(\log m)$ (got rid of f)
- Total # of iterations $O(L \log m)$

Parallel Algorithm: Reverse Delete

- Complicated!
- $\mu = \tau^2$
- $\#$ of iterations $O(L^2)$



Open problems in parallel setting

- Approximation Ratio: $O(\tau^2) \rightarrow O(\tau)$
- $\#$ of iterations: $O(L^2) \rightarrow O(L)$
- Find algorithms for τ-SNC systems beyond logarithmic length

Thank you!